# A CONTINUOUS MARKOV CHAIN MODEL FOR ESTIMATING ECONOMIC RETAIL MORTGAGE PORTFOLIO SIZE

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## ABSTRACT

In this study, a continuous Markov Chain model is used for modeling the size of retail loans in prepayment, past due, and default states. Prepayment and past due states describe the payment status of a loan. The default state is defined as charge-off on the loan due to bankruptcy, death, or other causes. As such, the model uses the economic status of the loan, rather than the accounting assets status. On the other hand, the book amount of a bank's credit portfolio on its financial statement seldom reflects its real economic status due to the nature of bookkeeping, which only provides a static snap-shot of a bank's operation result. Furthermore, the book amount fails to give the management a true picture of the portfolio pool, which is a function of its contraction and is based on past due rate, default rate, and prepayment rate. To remedy this situation, a stochastic model based on a continuous time Markov Chain is used to analyze contraction and extension, which give a true economic picture of a bank's credit portfolio and, thus in turn, facilitate the pricing of the bank's securities.

## INTRODUCTION

A bank portfolio's real volume, which is measured as the notional volume considering the risks associated with it, is of crucial interest to risk professionals because it could provide a buffer against credit risk events and is one of the basic criteria for evaluating credit operation performance. As a result, analysis of a portfolio's volume should be consistent with the definition of economic assets. The portfolio's assets listed on a bank's book are not necessarily of the same volume after taking credit risk into account. Thus, a dynamic snapshot of the portfolio's true status (taking quantity, as well as quality, into consideration) is crucial for daily credit risk management. Quality refers to the ability of a portfolio to stay in or transit back to the current state (that is a loan is neither past due nor prepaid).

Hence, the economic asset is the true amount of the portfolio that is generating interest income and providing the shield against some credit catastrophies. The recent sub-prime mortgage crisis is a verification of the above dual characteristics of the credit analysis.

The structure of this study is organized as follows: Section one provides a review of studies using Markov models to perform credit analysis. Section two gives a theoretical derivation of the portfolio economic assets model based on continuous Markov chains. Section three provides an empirical application of the model and section four concludes the study and proposes future studies.

**LITERATURE REVIEW**

Stochastic process methodology has played a significant role in credit asset management. As summarized by White (1993), Markov decision models have been mainly used in 18 areas, including (1) Finance and Investment, (2) Insurance, and (3) Credit Analysis. Of the 98 papers discussed by White, 9 papers relate to finance and investment, 2 to insurance, and 2 to credit analysis. This survey is by no means comprehensive, but it reveals the fact that Markov chains have been used extensively to analyze financial data.

Cyert, Davison, and Thompson (1962) developed a finite stationary Markov chain model to predict uncollectible amounts (receivables) in each of the past due category. This classic model is referred to as the CDT model. The states of the chain ($S_j$, $j = 0,1,2,...,J$) were defined as normal payment, past due, and bad-debt states. The probability $P_{ij}$ of a dollar in state $i$ at time t transiting to state $j$ at time t+1 is given as

$$P_{ij} = \frac{B_{ij}}{\sum_{m=0}^{J} B_{im}} \tag{1}$$

where $B_{ij}$ is the amount in state $S_j$ at time t+1 which came from state $S_i$ in the previous period. $S_t = S_o Q^t$ is the vector whose $j$ th component is the amount outstanding for the $j$ th past due category at the beginning of the $t$ th period for $t = 1,2,...,$. Here, $Q$ is a sub-matrix, in the transition probability matrix $P_{ij} = \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$, which includes transition probabilities among the set of transient states.

The model of Cyert, Davison, and Thompson (1962) was criticized by Corcoran (1978) who claimed that the representiveness of the transition probability could be affected by the "dominancy of large accounts." Therefore, he suggested grouping the accounts according to their size, and then a transition matrix for each group was provided by an exponentially smoothed matrix: $A_j = 0.8T_j + (1-0.8)A_{j-1}$, where $A_j$ is an exponentially smoothed matrix for month $j$ and $T_j$ is the transition matrix for month $j$.

Kuelen, Spronk, and Corcoran (1981) published their study on the CDT model and claimed that there was a flaw in the model because it failed to consider the partial payments for accounts due. By using the "total balance method," CDT understated the collection, and thus overestimated bad debts. A simple remedy, other than model structure modification, was to treat the partially paid amount and the remainder balance separately. As a result, an exact agreement with total receipts and aging could be achieved.

According to Thompson (1965), one of two important related tests for a bank's credit asset from the lender's point of view is the possibility of the loan getting into trouble, which means the probability of being in a past-due or even charged-off state. Another test is the extent of loss in the case of being in trouble. This could mean two things: (1) the recovery from collateral in the case of being charged off, or (2) the ability for an individual to bring himself back on track. Also, in the same paper, Thompson provided evidence supporting his claim that the business cycle and the macroeconomic situation are probably the most significant factors affecting change in bank credit.

Raj, Kirkham, and Clarke (1979), by assuming that new customers behave in a way similar to existing customers, implemented a Markov Model:

$$y(t) = \sum y(t-1) p_{ij} + u(t), i, j = 1,2,..,r; t = 1,2,..,T \tag{2}$$

where $y(t)$ is the observed liability shares of deposit-taking institutes at time $t$, $p_{ij}$ is the Markov transition probability from state $i$ to state $j$, and $u(t)$ is a disturbance term to analyze the growth rates of Canadian deposit-taking institutes because of the implication of the Bank Act in 1967. The authors showed that the model was appropriate whenever growth was dominated by macro economic factors and technical innovations.

By taking economic factors into account, Grinold (1983) used a finite Markov chain model to analyze a firm's market value if the firm follows an optimal policy in state $(x, y)$ at time $t$, where $x$ is the condition of the firm, and $y$ is the condition of the overall economy. He assumed that a change in state is governed by a stationary transition function. For instance, if the state is $y(t-1)$ at time $t-1$, then it will be $y(t)$ at time $t$ with probability $\pi[y(t)y(t-1)]$. However, to calculate $V_i^k$, he used dynamic programming because direct computation could be very time-consuming.

Jarrow, Lando, and Trumbull (1997) applied continuous and discrete time Markov chains to describe the default behavior of zero-coupon bonds within a time interval $\eta_t (0 \le t \le \tau)$. Furthermore, the default state was defined as an absorbing state. Again, the purpose was to price the bond based on analysis of credit risk spread. Similar approaches have been adopted by Liebman (1972) and Zipkin (1993). Lieberman used a Markov chain to model decision-making for credit card application approval. The states of the chain were $n+1$ paid-up states and one default state. The model assumed that the amount of dollars moved from one state to another follows a Markov chain process. On the other hand, Zipkin adopted a simpler model of interest rate, based on a discrete-time, finite-state Markov chain, to evaluate mortgage-backed securities. Glennon and Nigro (2005) used the survival analysis approach to measure the default risk of a small business. They adopted the Cox Proportional Hazard model. By using a discrete-time hazard procedure, they found that the default risk peaked in the second year after initiation, increased during the medium-maturity season, and declined thereafter.

Hurd and Kuznetsov (2007) introduced a conditional Markov Chain model, based on stochastic factors such as the interest rate and recovery rate, for credit default analysis. Their model, as claimed by the authors, can provide computational effectiveness with increased flexibility.

A parsimonious and flexible multivariate Markov Chain model to capture the dependency of transitions of ratings of credit risky entities has been proposed by Ching, Siu, L-m Li, T. Li, and W-K Li (2007) with an effective and flexible algorithm. The model, as well as the algorithm, reduces considerably the number of parameters. Similarly, algorithm has been implemented by Siu, Ching, Fung, and Ng (2005) to achieve a better risk measurement for credit risk.

A continuous Markov Chain model and algorithm have been used by Crommelin and Eijnden (2006) for simulation of molecular dynamics and of atmospheric flows and by Donatelli, Haddad, and Sproston (2006) for the Continuous Stochastic Logic, or CSL, when the measure of interest depends on the execution path.

## CONTINUOUS TIME MARKOV MODEL

In the continuous time Markov Chain model, let $S_j$ denote a past due state , corresponding to the number of days past due. The loan normally requires monthly payment. If a loan is 30 days past due, denote it by state $S_1$. State $S_2$ refers to 60 days past due and state $S_3$ represents a loan that is more than 91 days past due. Let $R_3, R_4$ be the states of prepaid defined as $R_3 = (X_t - Y_t)/B_t > 50\%, R_4 = (X_t - Y_t)/B_t < 50\%$ , respectively, where $X_t$ is the actual payment at month $t$ and $Y_t$ is the scheduled payment at month $t$. One can see that state $R_3, R_4$ is defined as the extra payment over the scheduled payment, which measures how much of the loan balance (or size) has been made as a current one- time payment.

Classification of the states of the Markov Chain into past due and prepayment states as well as default states is presented in Table 1.

**TABLE 1. DEFINITIONS OF THE DIFFERENT STATES OF THE MARKOV CHAIN**

| Past Due and Prepayment States $S_j, j = 0,1,2,3$ | | Default States $R_k$ $R_k, k = 1,2,3,4$ | |
|---|---|---|---|
| $S_0$ | No more than 30 days past due | $R_1$ | Sold by Bank |

| | | | |
|---|---|---|---|
| $S_1$ | 31 days – 60 days past due | $R_2$ | All other charge-off reasons |
| $S_2$ | 61 days – 90 days past due | $R_3$ | Prepayment of more than 50% of the remaining loan |
| $S_3$ | More than 91 days past due | $R_4$ | Prepayment of less than 50% of the remaining loan |

The salient feature of this model is the evaluation of loan asset behavior over time, which is more informative than the traditional accounting financial reports.

First, we define the time interval to be $(0,t), t<\infty$. The transition intensities between different states can be defined as (Chiang, 1980):
$v_{ij}\Delta t$ = Pr {an individual in state $S_i$ at time $\tau$ will be in state $S_j$ at time $\tau+\Delta t$ }, where $i \neq j; i, j = 0,1,2,3; i \neq j$
$\mu_{ik}\Delta t$ = Pr {an individual in state $S_i$ at time $\tau$ will be in state $R_k$ at time $\tau+\Delta t$ }, where, $i = 0,1,2,3$ and $k$ refers to the default states, $k = 1,2,3,4$.

Furthermore, we assume that the intensities $v_{ij}$ and $\mu_{ik}$ are independent of time. Thus, we are concerned here with a time homogenous Markov Chain.

If an individual stays in its original state, its intensity is defined as $v_{ii} = -(v_{ij} + \sum_{k=1}^{4} u_{jk})$, $i \neq j, i, j = 0,1,2,3, k = 1,2,3,4$. By this definition, it is obvious that

$$1 + v_{ii}\Delta t = \Pr\{\text{an individual in state } S_i \text{ at time } \tau \text{ will be in state } S_i \text{ at time } \tau+\Delta t\}.$$

Within any single time interval, $\{\tau, \tau+\Delta t\}$, V is the prepayment and past due intensity matrix, while U is the default intensity matrix:

The matrix of transition intensities between the S-states (prepayment and past due states) is given by the V matrix in Figure 1. Also, the U matrix in Figure 1 represents the transition intensities from the S-states to the default states:

$$V = \begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array} \qquad U = \begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,3} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}$$

**FIGURE 1. TRANSITION INTENSITIES WITHIN THE S-STATES (V MATRIX), AND FROM THE S-STATES TO THE ABSORBING STATES (U MATRIX)**

Because $R_k$ is an absorbing state, there is no transition from an R to an S-state. Also, for a past due state, transition lies only between neighboring states. This result is obvious since within one month, a loan with no past due payment cannot have a two-month due payment.

## TRANSITION PROBABILITIES

Let $P_{ij}(\tau,t)$ = Pr {an individual in state $S_i$ at time $\tau$ will be in state $S_j$ at time $t$ }, $i, j = 0,1,2,3$. By definition, we have

$$\begin{aligned} P_{ij}(t,t+\Delta t) &= v_{\gamma j}(t)\Delta t \\ P_{jj}(t,t+\Delta t) &= 1 + v_{jj}(t)\Delta t \end{aligned} \quad (3)$$

$$P_{ij}(\tau,t+\Delta t) = P_{ij}(\tau,t)P_{jj}(t,t+\Delta t) + \sum_{\gamma\neq j} P_{i\gamma}(\tau,t)P_{\gamma j}(t,t+\Delta t) \quad . \quad (4)$$

By substituting Eq. (3) in Eq. (4) and rearranging, we have

$$\frac{P_{ij}(\tau,t+\Delta t) - P_{ij}(\tau,t)}{\Delta t} = P_{ij}(\tau,t)v_{jj}(t)\Delta t + \sum_{\gamma\neq j} P_{i\gamma}(\tau,t)v_{\gamma j}(t)$$

$$\Rightarrow \lim_{\Delta\to 0} \frac{P_{ij}(\tau,t+\Delta t) - P_{ij}(\tau,t)}{\Delta t} = \sum_{\gamma\neq j} P_{i\gamma}(\tau,t)v_{\gamma j}(t)$$

$$\Rightarrow \frac{\partial}{\partial t} \; P_{ij}(\tau,t) = \sum_{\gamma\neq j} P_{i\gamma}(\tau,t)v_{\gamma j}(t); i, j = 0,1,2,3 \quad (5)$$

Equation (5), the Kolmogorov Forward Differential Equation, and its solution is given (Chiang, 1980) as

$$P_{ij}(0,t) = \sum_{l=0}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m\neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l t}, i, j = 0,1,2,3 \quad (6)$$

Here, $A_{ij}'$ is the characteristic matrix of $V'$, the transpose of the intensity matrix $V$ , defined by

$$A_{ij}' = (\rho I - V'), \quad (7)$$

where $\rho_l$ = Eigenvalue of the intensity matrix $V$ .

For an individual in $S_i$ at time 0, let $e_{ij}(t)$ = the expected duration of stay in $S_j$ during the interval $(0,t)$, $j = 0,1,2,3$. In terms of our process, $e_{ij}(t)$ evaluates the expected duration of the loan before default occurs. This expected duration, $e_{ij}(t)$ , can be expressed (Chiang, 1980) as

$$e_{ij}(t) = \int_0^t P_{ij}(0,\pi)d\pi \tag{8}$$

$$e_{ij}(0,t) = \sum_{l=0}^{3} \frac{A_{ij}'(\rho_l)}{\prod_{\substack{j=0 \\ j \neq l}}^{3} (\rho_l - \rho_j)\rho_l} (e^{\rho_l t} - 1), i, j = 0,1,2,3 \tag{9}$$

## A MARKOV MODEL FOR ECONOMIC ASSETS ANALYSIS

In any time interval, the size (balance) of a portfolio is a function of contraction and extension. For the purpose of this study, contraction refers to any process that causes a reduction in credit assets. On the other hand, extension is defined as any process that causes an increase in portfolio size.

What makes a prepayment state absorbing is the fact that a prepayment cannot be deducted from the next scheduled payment. For normal operation, one expects the bank's credit assets to be in state $S_0$. The following three reasons validate the classification of prepayment as an absorbing state:

1. The prepaid loan amount (extra payment besides the scheduled normal payment) cannot serve as a buffer for future payment.
2. The prepaid loan amount cannot be refunded by the bank.
3. The prepaid loan amount reduces the overall portfolio balance permanently.
4. The prepaid loan amount cannot be used to calculate the interest payment for the next period of time.

Please note that, at any point in time, the state of no more than 30 days past due, $S_0$, refers to a health state, and we expect most of a bank's credit assets to stay in this state for normal operations.

The purpose of this Markov model is to analyze the portfolio size (or balance) for a bank within a time interval $(0,t)$. This fact implies the evaluation of the value after taking potential risks into account, instead of the accounting amount based on the bank's financial statement. The model can provide a true snap-shot at any given time $\xi$ within a time interval $(0,t)$ for the management, and also fundamental information for investors in a trading period interval $(0,t)$. The model has the following assumptions:

1. A transition an individual might make in the future is independent of those made in the past.
2. Individuals do not have equal probability of default, which depends on the specific debt structure, liquidity requirement, and risk taking ability.
3. The bank is under normal operation where the rate of approval of loan applications is assumed to follow a Poisson process.

For each $\tau, 0 \le \tau < t$, a change in the population size of each state $S_i$ during a single time interval $(\tau, \tau + \Delta t)$ occurs based on the following probabilities:
$\lambda_i \Delta t$ = Probability that state $S_i$ ( $i = 0,1,2,3$ ) increases by 1 during a single time interval $(\tau, \tau + \Delta t)$. It is assumed that λ (the intensity of the Poisson process) is independent of time.
$v_{ij}(\tau)\Delta t$ = Pr {one individual will move from state $S_i$ to state $S_j$ during the time interval $(\tau, \tau + \Delta t), i, j = 0,1,2,3$ }
$\mu_{ik}(\tau)\Delta t$ = Pr {an individual will move from a non-absorbing state $S_i$ to one of the absorbing state $R_k$ during $(\tau, \tau + \Delta t), i = 0,1,2,3, k = 1,2,3,4$ }.

The intensity $v_{ij}$ that an individual stays in its original state in the time interval $(\tau, \tau + \Delta t)$, is defined as $v_{ii} = -(v_{ij} + \sum_{k=1}^{4} u_{jk}), i \ne j, i, j = 0,1,2,3, k = 1,2,3,4$. By this definition, it is obvious that $1 + v_{ii}\Delta t$ = Pr {an individual in state $S_i$ at time $\tau$ will be state $S_i$ at time $\tau + \Delta t$ }. Within any single time interval, { $\tau + \Delta t$ }, V is the non-absorbing intensity matrix, while U is the absorbing intensity matrix as shown in Figure 1.

Due to the fact that the R states are absorbing, there is no transition from U to V or among the R-states in U.

It is obvious that an increase in a portfolio's size within a small time interval { $\tau$ , $\tau + \Delta t$ } could be regarded as the result of only the migration process. As a result, the portfolio size at any given time $t$ can be expressed as

$$X(t) = Y(t) + Z(t) \tag{10}$$

$$X(t) = \begin{pmatrix} X_0(t) \\ X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix}, Y(t) = \begin{pmatrix} Y_0(t) \\ Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix}, Z(t) = \begin{pmatrix} Z_0(t) \\ Z_1(t) \\ Z_2(t) \\ Z_3(t) \end{pmatrix}$$

where $X_i(t), i = 0,1,2,3$ is defined as the portfolio size in each of the states, $s_0, s_1, s_2, s_3$ at time $t$. $Y_i(t), i = 0,1,2,3$ refers to the portfolio size in state i at time t that survived from the original portfolio in state $i$ ( $i = 0,1,2,3$ ) at time zero, and $Z_i(t), i = 0,1,2,3$ stands for the portfolio size in state i at time t as a result of immigration during the interval $(0, t)$. One can argue that $Y_i(t)$ is affected both by contraction and extension, while $Z_i(t)$, a pure incremental factor, is merely an extension process.

The extension process is composed of:

1. Immigration or increase in the portfolio size because of approved new applications for

a particular loan offered by a bank.
2. Birth or increase in the value of the original portfolio at time 0 because of the passage of time.

For simplicity, however, we will consider only immigration in this study. That is, we consider approval of a new loan as the only factor that plays a role in the extension process. On the other hand, the contraction process is triggered by three factors:

1. Prepayment, or the additional payment for a loan besides the schedule payment, reduces the portfolio size prematurely.
2. Default, causing the elimination of the default loan amount from the portfolio, is considered as another contraction force.
3. Transition, an individual moving from an original state to another state.

Thus, letting $m_i$ be the portfolio size at state $i, i = 0,1,2,3$, at any time $\tau, 0 \le \tau \le t$ the expected portfolio value is given by

$$E[X_j(t)] = \sum_{i=0}^{3} m_i p_{ij}(0,t) + q_j(t), i, j = 0,1,2,3 \qquad (11)$$

and, the variance is given by

$$V[X_j(t)] = \sum_{i=0}^{3} m_i p_{ij}(0,t)[1 - p_{ij}(0,t)] + q_j(t), i, j = 0,1,2,3 \qquad (12)$$

where, $p_{ij}(0,t)$ is the probability of being in state $j$ at time $t$ given that the process was in state $i$ at time zero.

$$p_{ij}(0,t) = \sum_{l=0}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m \neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l t}, i, j = 0,1,2,3 \qquad (13)$$

as obtained from the solution to the Kolmogorov Forward Differential Equation:

$$\frac{\partial}{\partial t} p_{ij}(\tau,t) = \sum_{\gamma \neq j} p_{i\gamma}(\tau,t) v_{\gamma j}(t), i, j = 0,1,2,3 \qquad (14)$$

Also, $q_j(t)$ is the expected portfolio size in state $S_j$ at time $t$, and is given by

$$q_j(t) = \sum_{i=0}^{3} \int_0^t \lambda_i \cdot p_{ij}(\tau,t) d\tau$$

$$
\begin{aligned}
&= \sum_{i=0}^{3}\int_0^t \lambda_i \cdot \sum_{l=0}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=0\\ m\neq l}}^{3}(\rho_l-\rho_m)} e^{\rho_l(t-\tau)}d\tau, \\
&= \sum_{i=0}^{3}\sum_{l=0}^{3}\lambda_i \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=0\\ m\neq l}}^{3}(\rho_l-\rho_m)}(e^{\rho_l t}-1), i, j = 0,1,2,3
\end{aligned}
\qquad (15)
$$

Here, $\lambda_i$ is the immigration rate to state $S_i$ and $A_{ij}'$ is the $ij^{th}$ element of the characteristic matrix of $V'$, the transpose of the intensity matrix $V$, defined by

$$A_{ij}' = (\rho I - V'), \qquad (16)$$

where $\rho_l =$ eigenvalue of the intensity matrix $V$.

## APPLICATION

Data were provided by a local bank in Ohio, operating in Ohio, Michigan, Kentucky, and Indiana. By using its monthly paid retail mortgage loan for 16 consecutive months, from April 2005 to September 2006, one can demonstrate the applicability of the continuous time model.

For a continuous-time Markov Chain, an element $v_{ij}$ of the transition matrix $V$, is given by the following equation:

$$v_{ij} = \frac{d}{dt} P_{ij}(c_{ijt}, t)\,|_{t=0}, i \neq j, i = 0,1,2,3, t = 1,2,3,...16, \qquad (17)$$

where $P_{ij}(c_{ijt}, t)$ stands for the 5$^{th}$-order polynomial used to fit the observed transition probabilities from the data over time. $c_{ijt}, i, j = 0,1,2,3 \ \ , t = 1,2,3,...16$ The polynomials are approximated by the Lagrange numerical method. For instance, using MATLAB 7.0 ® Release 14.

The diagonal elements of the intensity matrix $V$ and U are given by

$$v_{ii} = -(v_{ij} + \sum_{k=1}^{4} u_{ik}), i \neq j, i, j = 0,1,2,3, k = 1,2,3,4 \qquad (18)$$

where $u_{ik} = \frac{d}{dt} P_{ik}(r_{ikt}, t)\,|_{t=0}, i = 0,1,2,3, t = 1,2,3,...16$. Thus, we obtained the following $V, U$ transition intensity matrices as presented in Figure 2:

$$V = \begin{bmatrix} -0.9962 & 0.1141 & 0.0909 & 0 \\ 0.2513 & -1.0263 & 0.2986 & 0.1849 \\ 0.0526 & 0.3008 & -0.9606 & 0.3426 \\ 0.0904 & 0.1416 & 0.3512 & -0.9231 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.0126 & 0.0125 & 0 & 0 \\ 0 & 0.0245 & 0.1214 & 0 \\ 0 & 0.1745 & 0.0215 & 0 \\ 0 & 0.1842 & 0.0047 & 0 \end{bmatrix}$$

**FIGURE 2. INTENSITY MATRICES V AND U**

Hence, one can estimate, from Eq (6) and Eq (9), the transition probability matrix $P_{ij}(0,1)$ and the expected duration of stay in state $j$ (given that the process started in state $i$) during the interval $(0,1)$, $e_{ij}(1), i, j = 0,1,2,3$. These are given in Figure 3:

$$P_{ij}(0,1) = \begin{bmatrix} 0.7842 & 0.1141 & 0.0624 & 0.0774 \\ 0.1456 & 0.5748 & 0.0944 & 0.0641 \\ 0.0874 & 0.1457 & 0.5247 & 0.0451 \\ 0.1047 & 0.0347 & 0.0974 & 0.2473 \end{bmatrix}$$

$$e_{ij}(0,1) = \begin{bmatrix} 0.8947 & 0.0784 & 0.0142 & 0 \\ 0.1249 & 0.5006 & 0.1471 & 0.0009 \\ 0.0972 & 0.1001 & 0.4478 & 0.4781 \\ 0.0019 & 0.0133 & 0.1404 & 0.2874 \end{bmatrix}$$

**FIGURE 3. TRANSITION PROBABILITY MATRIX AND EXPECTED DURATIONS OF STAY IN STATE J (STARTING IN STATE I) IN THE INTERVAL (0, 1)**

For instance, $P_{2,1}(0,1) = 0.1456$ represents the probability that a loan in the 2-month past due state will transit to the 1-month past due state during the time interval $(0,1)$. On the other hand, $e_{2,1}(1) = 0.1249$ represents the mean time in months of stay in the 1-month past due state (given that the loan started in the 2-month past due state at t = 0) in the time interval $(0,1)$. The numbers in the matrix $e_{ij}(0,1)$ represent the percentage of time the process stays in a state in the unit interval (0, 1). Therefore, $e_{ij}(0,30)$ is the estimated duration for each of the states in the interval of (0, 30), or one month.

## ECONOMIC ASSETS

In this subsection, we will use the model to approximate the stochastic retail mortgages portfolio size of the Ohio bank. Let $X(t)$ be the total stochastic retail mortgage portfolio size at time $t$. Its expected value can be expressed as

$$E[X(t)] = \sum_{j=0}^{3} E[X_j(t)], j = 0,1,2,3 \quad , \tag{19}$$

where $E[X_j(t)]$, the expected portfolio size belonging to state $j$, is given in Eq (11). The following set of equations was used in applying the algorithm provided by MathCAD.

$$\begin{aligned} E[X_j(t)] &= \sum_{i=0}^{3} m_i p_{ij}(0,t) + \sum_{i=0}^{3} \int_0^t \lambda_i \cdot p_{ij}(\tau,t) d\tau, \\ p_{ij}(0,t) &= \sum_{l=0}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=0 \\ m\neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l t}, i, j = 0,1,2,3 \end{aligned} \tag{20}$$

Where $m_i$ is the retail mortgage portfolio size belonging to state $i$ in thousands of dollars at time 0 or April 2005, given in Fig 4, and $\lambda_t$ is the immigration rate at time $t$, which is estimated by Eq. (21) below.

Using the bank database, we estimated the $M = \{m_i\}$ vector as shown in Figure 4, in thousands of dollars:

$$\begin{array}{ccccc} & S_0 & S_1 & S_2 & S_3 \\ m = & (68428.91 & 292.79 & 267.11 & 62.31)^T \end{array}$$

**FIGURE 4. RETAIL MORTGAGE DISTRIBUTIONS IN THOUSANDS OF DOLLARS AT TIME 0**

The estimation of the immigration rate is given by the following method. For simplicity, we assume that the immigration intensity or increment rate is homogeneous over time ( $\lambda_i(t) = \lambda_i$ ). Let $f_{\lambda_i}$ be the 5th order polynomial function for $\lambda_i$ from the one step immigration dollar amount at time $t$, $i_t$. Thus, by taking the first-order derivative of the function $f_{\lambda_i}$, evaluated at time t = 0, we obtain the immigration intensity

$$\begin{aligned} \lambda_i &= \frac{df_i(\lambda_{t,i})}{dt} |_{t=0}, t = 1,2,...,16, i = 0,1,2,3 \\ \lambda_{t,i} &= \left\{ \begin{array}{cccc} t_1 & t_2 & & t_{16} \\ \begin{pmatrix} \lambda_{1,0} \\ \cdots \\ \lambda_{1,3} \end{pmatrix}, & \begin{pmatrix} \lambda_{2,0} \\ \cdots \\ \lambda_{2,3} \end{pmatrix}, & \ldots, & \begin{pmatrix} \lambda_{16,0} \\ \cdots \\ \lambda_{16,3} \end{pmatrix} \end{array} \right\} \end{aligned} \tag{21}$$

where $\lambda_{t,i}$ is the retail mortgage immigration rate between period $t$ and period $t-1$ in state $i$.

Using the same approach as in figure 2, we estimated the transition intensity matrices, V and U as shown in Figure 5:

$$V = \begin{bmatrix} -0.9674 & 0.1584 & 0.1002 & 0 \\ 0.3047 & -0.9912 & 0.2784 & 0.1748 \\ 0.0614 & 0.2947 & -0.7843 & 0.1978 \\ 0.1047 & 0.1687 & 0.3314 & -0.9047 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.0784 & 0.0087 & 0 & 0 \\ 0 & 0.0144 & 0.0014 & 0.0241 \\ 0 & 0.1547 & 0 & 0.0078 \\ 0 & 0.1574 & 0 & 0.0003 \end{bmatrix}$$

**FIGURE 5. TRANSITION INTENSITY MATRICES FOR STOCHASTIC ASSETS**

Please note that in figure 2, the matrix V and U were calculated by observing the number of accounts transited, while in figure 5, we used the observation of transited loan amounts. We believe that the observation method in figure 5 would fit better in the study of economic portfolio size.

Letting $InTran(0,t)$ be the portfolio assets distribution from internal transition and immigration and $ExTran(0,t)$ be the loan balance transited from a non-absorbing state to any absorbing state, we have the following results as shown in Figure 6

$$\begin{array}{ccccc} & S_0 & S_1 & S_2 & S_3 \\ InTran(0,t) = ( & 3478.21 & 223.7 & 312.09 & 107.45)^T \end{array}$$

$$\begin{array}{ccccc} & R_1 & R_2 & R_3 & R_4 \\ ExTran(0,t) = ( & 20.14 & 35.14 & 85.87 & 102.8)^T \end{array}$$

**FIGURE 6. INTERNAL ASSETS AND IMMIGRATED ASSETS DISTRIBUTIONS OVER STATES**

As one month is the usual measure period of banks, by letting $t = 30$, we can estimate $A_{monthly}$, the stochastic assets of the monthly paid retail mortgage assets in thousands of dollars, by the following equation:

$$\begin{aligned} A_{monthly} &= \sum_{i=0}^{3} [InTran_i(0,30) - ExTran_i(0,30)] \\ &= \$3,877.5 \end{aligned} \tag{22}$$

## CONCLUSION

The above models will allow the bank management to analyze its loans characteristics in any reasonable interval. The following matrices are obtained by letting $t = 30$ in Figure 7:

$$P(0,30)=\begin{bmatrix} 0.7841 & 0.1765 & 0.0014 & 0.0784 \\ 0.2147 & 0.6574 & 0.2011 & 0.0991 \\ 0.0874 & 0.1043 & 0.5471 & 0.2457 \\ 0.0741 & 0.0471 & 0.1140 & 0.3284 \end{bmatrix}$$

$$e(0,30)=\begin{bmatrix} 1.8715 & 0.0147 & 0.0782 & 0.0011 \\ 0.1478 & 0.5871 & 0.2478 & 0.0741 \\ 0.2144 & 0.3212 & 0.5478 & 0.1247 \\ 0.0012 & 0.0314 & 0.5421 & 0.4478 \end{bmatrix}$$

**FIGURE 7. TRANSITION PROBABILITY $(0, 30)$**

The value for $P_{3,3}(0,30)$ means that the probability of staying in a 3-month past due state for 30 days is 0.3284. Also, $e_{3,3}(30)=0.4478$ tells us that, during the interval $(0,30)$, staying in 3-month past due state is only 0.86326 unit of time. Furthermore, one can see that a small value for $P_{i,j}(0,t)$ is usually accompanied by a small value for $e_{i,j}(t)$, which is what one expects based on banking experience.

As can be seen, there is a large difference between the retail mortgage's book amount on the bank's financial statement and the estimated stochastic amount which take into consideration the prepayment, past due, and default after one month. The latter is often of most interest to the outside investors because this is the real assets amount that could be used to buffer the liability due to the customer's deposit. In most cases, it could be used to evaluate the bank's operation efficiency as well as its bankruptcy potential.

The above model used only the occurrence frequencies of each state and did not consider the loan asset. Future work may consider the loan asset in developing a model for risk management in the banking industry.

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